Wigner Crystallization: A Brief Overview of the Theoretical and Experimental Development from its Inception to the Present

1 Introduction

Wigner crystallization, named after the physicist Eugene Wigner, is an exotic phenomenon seen in electron gases living on a periodic lattice. At sufficiently low densities, electrons in such an environment will have very little correlation with their neighbors. As the potential energy term outcompetes the kinetic energy term at such inter-cell distances because of r^{-1} vs. r^{-2} dependence, the electrons on the lattice will settle to the bottom of the respective potential "dimples" they reside in, essentially localizing into a crystalline structure. In this paper, we trace the development of the subject through select papers that are representative of the progress in the field, from Wigner's original proposition to later numerical explorations, and to the more recent experimental observations on surfaces of liquid helium, structures under strong external magnetic fields, and other two-dimensional structures such as twisted bilayer graphene.

2 Theoretical development

2.1 Wigner's theory and early interpretations

In the original paper written in 1934 [1], Wigner considered an electron gas in a periodic potential. By incorporating Coulomb interactions between electrons into the calculations with the use of Hartree-Fock wavefunctions, Wigner found that for regimes where the electronic density is very low (where his approximations are valid), the electron-electron correlation energies diminish as r_s, the inter-electron distance, is increased. That can be explained as follows [2]. The energy of particles on a lattice depends on two terms: the Coulombic potential and kinetic energy. The Coulomb interaction of ions on a lattice increases quickly with increasing density, and they soon overwhelm contributions from the kinetic term. As a result, positive ions tend to be localized. The

case for electrons, on the other hand, cannot be more different. In the case of 2D electron systems where the electron density is $n_e = 1/\pi r_0^2$, the mean potential energy per electron is given by

$$U_c = \frac{e^2}{r_0} = \frac{2}{r_s}[Ry],$$

where $1\,\mathrm{Ry} = me^4/w\hbar^2 = 13.6\,\mathrm{eV}$ is the Rydberg constant and $r_s = r_0/a_B$ is the ratio of the atomic radius and the Bohr radius, $\hbar^2/me^2$. The average kinetic energy of a 2D electron system assuming a simple electron spectrum $\epsilon_k \propto k^2$, on the other hand, is written as

$$K_e = \frac{\epsilon_F}{2} = \frac{\hbar^2}{2m_e r_0^2} = \frac{1}{r_s^2}[Ry].$$

The r^{-1} vs. r^{-2} dependence suppresses the kinetic term compared to the Coulomb term when $r_s \gg 1$, giving rise to the localization of electrons. This result led Wigner to postulate the electrons in the ground state of such systems will form a perfect lattice—a so called "electron solid".

This state of matter is not limited to systems composed of electrons, however. In a paper written in 1967 [3], Van Horn postulated that if electrons can crystallize under the right circumstances, protons can certainly perform the same feat if given the right environment. He noted that in the core of white dwarfs, the density of electron is too high for Coulomb interactions to have any significant impact on the total energy, and as such, electrons near the core of white dwarfs will be uniformly delocalized, forming a neutralizing background charge. Protons suspended in this electron gas, on the other hand, will have a crystallization density that is substantially higher than the density at which pycnonuclear reactions take place and consume them. In effect, if the star is composed of hydrogen, the naked protons at its cold core that result from pressure ionization will crystallize at all densities low enough that nuclear reactions can be ignored. This is the charge and mass interchanged version of the kind of system Wigner contemplated.

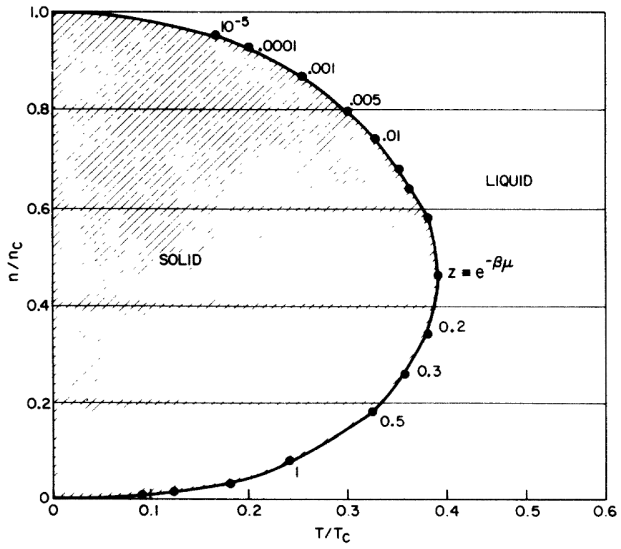

Figure 1: Parametrized phase diagram of the two-dimensional Coulombic system. [4]

2.2 Early assessments of the phase diagram

One of the earliest numerical studies of the Wigner crystal was done by D. Ceperley. In his 1978 paper [5], Ceperley used variational Monte Carlo coupled with a random-phase approximation pseudo-potential to solve for the interaction energies of an electron gas in two and three dimensions. The results of his group, when coupled with finite-size scaling, showed that at low densities and low temperatures, the electron gas crystallizes into what Wigner predicted in his original papers. When densities reach $r_s \leq 67 \pm 5$ in 3D and $r_s \leq 33 \pm 2$ in 2D, the crystal melts and becomes a polarized electron fluid. At higher densities still, when r_s reaches around 26 ± 5 in 3D and 13 ± 2 in 2D, the polarized electron fluid become unpolarized. The total energy of the Wigner state on a 2D hexagonal lattice was estimated in a study a year later to be

$$\varepsilon_{\mathrm{WS}} = -\frac{2.212}{r_s} + \frac{1.628}{r_s^{3/2}}[Ry],$$

where the negative sign of the first term comes from interaction with the positive background [6].

In a follow-up study in 1980 [7], the Ceperley group did a stochastic simulation of the same model in three dimensions at zero temperature. Their calculations, which covered $0 \leq r_s \leq 200$,

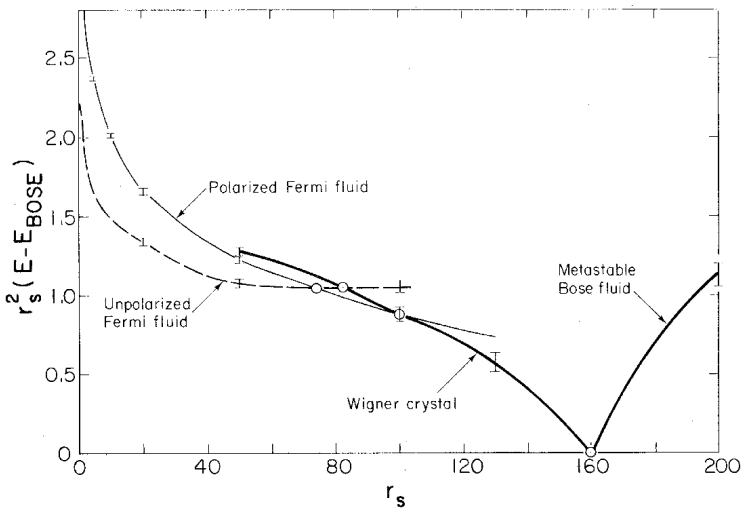

Figure 2: The energy of the four phases studied relative to that of the lowest boson state time r_s in rydbergs vs. r_s in Bohr radii. Below $r_s = 160$ the Bose fluid is the most stable phases, while above, the Wigner crystal is most stable. The energies of the polarized and unpolarized Fermi fluid are seen to intersect at $r_s = 75$. The polarized (ferromagnetic) Fermi fluid is stable between $r_s = 75$ and $r_s = 100$, the Fermi Wigner crystal above $r_s = 100$, and the normal paramagnetic Fermi fluid below $r_s = 75$. [7]

uncovered four quantum phases. In the range $0 \leq r_s \leq 75$, they found that the unpolarized Fermi fluid phase is the most energetically favorable, whereas the polarized Fermi fluid phase is favored, albeit only slightly, in $75 \leq r_s \leq 100$. Wigner crystallization occurs in $100 \leq r_s \leq 160$. Above $r_s \approx 160$, the metastable Bose fluid is the most stable phase.

A subsequent study in 1989 [8], using the more accurate fixed-node diffusion Monte Carlo simulation on larger systems. The authors adjusted the critical density of Wigner crystallization to $r_s = 37 \pm 5$. This newer study also found the normal fluid state is preferred over the fully polarized state at higher densities (lower r_s values), despite the latter is energetically competitive.

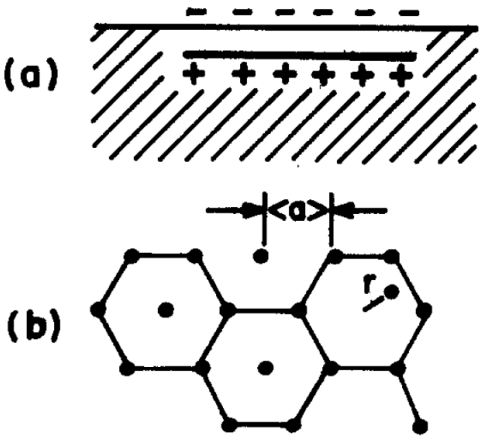

Figure 3: (a) Electrons on the surface of liquid helium confined to a limited area by a metal plate below the surface bearing an equal amount of positive charge. (b) Close-packed hexagonal lattice. [9]

3 Experimental observations and beyond

3.1 First concrete evidence of a Wigner transition

For a long time, experimental observation of Wigner crystallization was impeded by the required large inter-electronic distances. The situation changed when an experimental group at Princeton serendipitously discovered a way to achieve sufficiently low electronic densities when trying to measure the lifetime of electrons in surface states of liquid ^{4}He [9, 10]. The experimental setup they proposed consisted of a film of liquid ^{4}He with electrons deposited on its surface through corona discharge.

The first concrete experimental observation of Wigner crystallization was done in 1979 by a group at the Bell labs [11, 12]. This group, taking up on the proposal of the Crandall group, performed a measurement of a charged surface of liquid helium. Before the experiment, the group expected to observe standing capillary waves (ripplons), since when electrons crystallize on the surface of liquid helium, they could be driven up and down against the liquid helium surface with a perpendicular radio-frequency electric field and produce detectable ripplons. What was observed, however, was both vertical and horizontal resonances (coupled plasmon–ripplons). The results

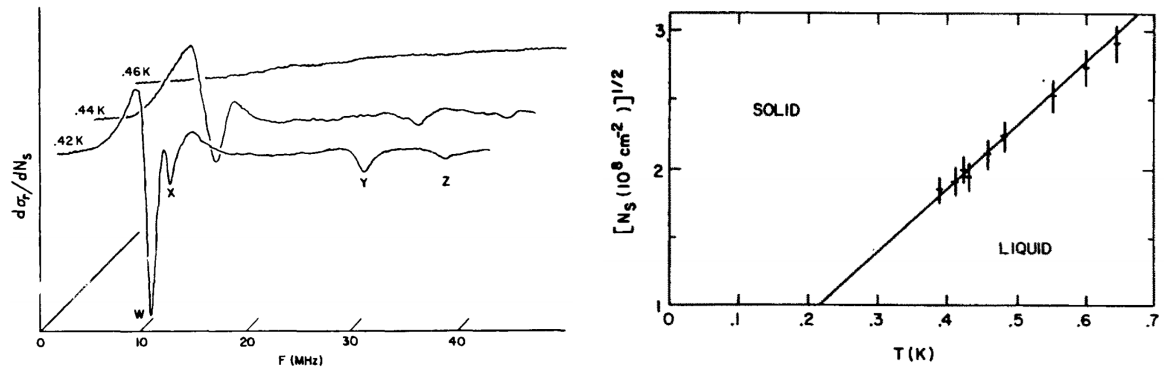

Figure 4: (Left) Experimental traces showing coupled plasmon–ripplon resonances at an areal density of $\approx 4.4 \times 10^8$electron/cm^2. These resonances disappear when the electron crystal melts at $T = 0.457$ K. (Right) The data points denote the melting temperatures measured at various values of the electron areal density, N_S. Along the line the quantity Γ has a value 131. [11]

were dramatic nonetheless, showing abrupt appearance of long-range ordering as temperature drops below 0.457 K, indicating crystallization.

3.2 Magnetically induced Wigner solids

Not long after Wigner crystallization was being directly observed on liquid helium surfaces, researchers were looking for other ways of observing it. An experimental group in France proposed in 1988 [13] that Wigner crystallization could be induced by an external magnetic field. Under an external magnetic field, electrons are confined by the Lorentz force to an area proportional to $\phi_0/H = 2\pi l_H^2$ and takes on the Landau energies $(n + \frac{1}{2})\hbar\omega_c$, where $\hbar\omega_c = \hbar eH/mc$ is the cyclotron energy. In this picture, on average, each Landau level is $\nu = 2(l_H/a)^2$-fold degenerate. When ν drops below 1, however, there is plenty of spatial freedom for each electron on the lattice, which makes Coulomb-induced correlations much more favorable. For a system with a filling factor below $\nu \approx \frac{1}{3}$, some crystal-like structure was expected. Several groups subsequently succeeded in inducing Wigner crystallization on GaAs/GaAlAs crystal with a filling factor of $\nu = \frac{1}{5}$. Among the many experiments performed, the French group probed the GaAs/GaAlAs crystals with electric fields with varying frequencies, to which the target showed resonances to frequencies corresponding to specific fractional wave vectors, indicating long-range order.

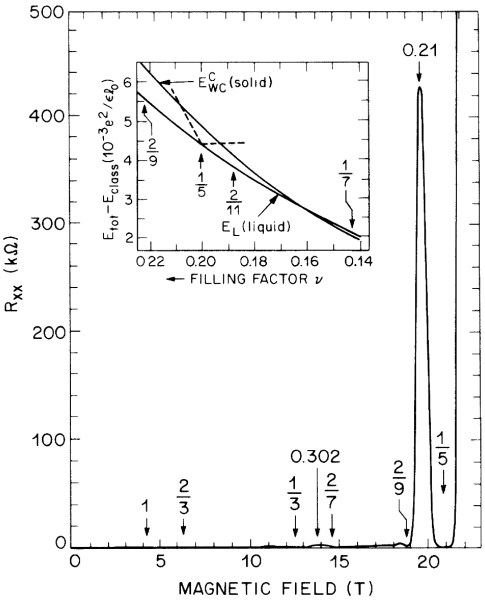

Figure 5: Diagonal resistance R_{xx} vs. magnetic field at $T \approx 90\,\mathrm{mK}$. Data area taken on a square sample so that $\rho_{xx} = \alpha R_{xx}$ with $\alpha \sim 1$. All FQHE features at lower magnetic field are well developed but practically invisible on this scale. Inset: Result of a calculation for the total energy per flux quantum of the solid (E^c_{WC}) and interpolated $1/m$ quantum liquids (E_L) as a function of filling factor. The dashed lines represent the cusp in the total energy of the liquid at $\nu = \frac{1}{5}$. Its extrapolation intersects the solid at $\nu \sim 0.21$ and 0.19 suggesting *two* phase transitions from quantum liquid to solid around $\nu = \frac{1}{5}$. [14]

Another important phenomenon associated with 2D systems that exhibit Wigner crystallization under a magnetic field is the fractional quantum Hall effect. Discovered in 1981 by Tsui and Störmer, the fractional quantum Hall effect is the quantized analogue of the classical Hall effect: instead of being a linear function of H, resistivity takes values $\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$ where $\nu \in \mathbb{Q}$ is some fraction [15]. This is precisely what a group at MIT found on top of the existence of Wigner crystallization [14].

3.3 Double-layer materials

The studies thus far have shown that Wigner crystallization does not occur under zero external magnetic field unless the density is very low—densities low enough that made direct observation of the phenomenon very difficult. It then came to the realization of the group of Świerkowski that

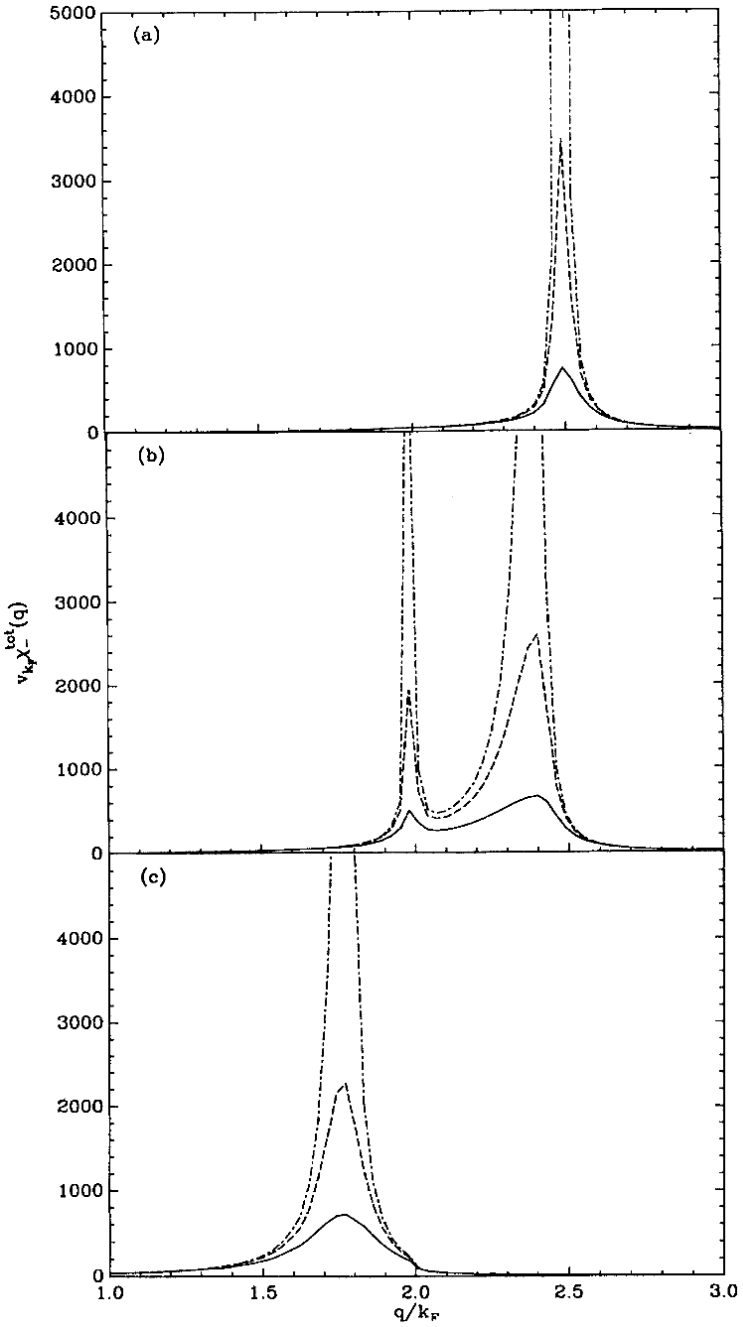

Figure 6: Static response function $\chi^{\rm tot}(q)$ for two layers. (a) Plots for density $r_s = 25$ and three different layer spacings, $a/a_B^* = 14.90$, 14.75, 14.71 (solid, dashed, and dot-dashed lines, respectively). Transition to the Wigner crystal occurs at critical layer spacing $a_c/a_B^* = 14.7$. (b) Similar to (a), but with $r_s = 20$ and $a/a_B^* = 9.65$, 9.57, and 9.55. Transition to the Wigner crystal at $a_c/a_B^* = 0.5$. (c) Similar to (a), but with $r_s = 10$ and $a/a_B^* = 3.26$, 3.245, and 3.24. [16]

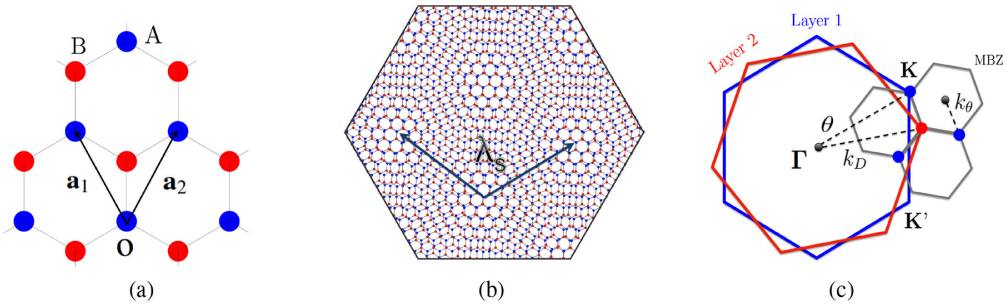

Figure 7: (1) (Real space) layer 1 of the TBLG system; (b) hexagonal (moiré) lattice with lattice constant λ_s formed by twisting (~10°) a layer of graphene relative to another; (c) (k-space) the corresponding moiré or mini Brillouin zone (MBZ). [18]

layers of 2D lattices could interact to make the Wigner crystal phase happen at much a much higher density, all without the presence of an external field [16]. In their numerical study, two layers of Wigner lattices are brought to proximity while maintaining a substantial distance between the layers such that the tunneling probability remains negligible. Coulomb interactions between the layers then contributes to the potential term in the electronic energy and facilitates the dominance of the potential term at electronic densities higher than $r_s \approx 37$. In fact, using Monte Carlo simulations, the group demonstrated that Wigner crystallization can still occur at densities a factor of 3 higher than that of a single isolated layer. A second numerical study in 1996 [17] showed that inter-layer interactions not only foster Wigner crystallization, but also work to melt the Wigner crystal and restore a liquid phase when the inter-layer distance become comparable to the lattice constant.

More recently, twisted bilayer graphene (TBLG) have gained attention for their superconducting state that is reminiscent of the copper oxide materials. TBLG systems are formed by first stacking two sheets of graphene in an AB configuration then rotate them through a certain A_1B_1 point, as shown in Fig 7. As the layers are rotated against one another, right around ~1° (the "magic" angle), the system transitions from being insulating to superconducting. It was argued that the insulating state was a Mott state, but a recent study [18] disputes the conclusion. In a twisted bilayer structure, the physics is found to be dominated by moirë supercells, which is an emergent periodic superstructure that forms when the two neighboring lattices are not perfectly aligned, as shown in Fig. 7. Here the inter-cell distance is accounted not by the lattice constants of the primitive

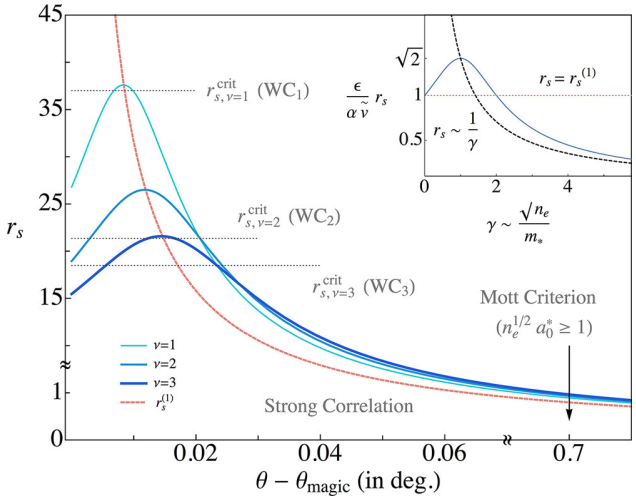

Figure 8: Body of the figure contains the twist-angle dependence of r_s. The blue curves are increasingly darker (or thicker) with increasing filling fraction. The dotted lines denote the critical r_s required to form the WCs for a given ν, $r_{s,\nu}^{\rm crit} = 37,\ 37/\sqrt{3}$, and $37/2$ for $\nu = 1,\ 2$, and 3, respectively. [18]

cells but by the lattice constant of the moirë supercell, $\lambda_s = a/2\sin(\theta/2) \approx a/\theta$, where a is the lattice constant of the primitive cell. At ~1°, this structure forms an effective triangular lattice with a unit cell that encloses about 13,000 carbon atoms. At electron densities that have been probed experimentally, only the two lowest bands are physically important. These two bands can house 4 electrons, having a supercell electron density

$$n_s = \frac{32\sin^2(\theta/2)}{\sqrt{3}a^2} \approx (\theta^\circ)^2 2.32\times 10^{16} e^-/\mathrm{m}^2,$$

which is far below the density required for Wigner crystallization to occur. Another fact that points toward Wigner crystallization is that the insulating states are found to be very sensitive to external magnetic fields, which is not typical in Mott systems, but as previous experiments have shown, highly relevant in the case of a Wigner crystal. Through numerical calculations, Padhi et al. showed that a Wigner crystallization should occur for $\frac{1}{4}$ filling on a triangular lattice. This is yet to be confirmed experimentally, but it nonetheless shows the relevance of Wigner crystallization in a modern context.

About the physicist [19]

Eugene Paul Wigner (Nov, 17, 1902 – Jan. 1, 1995) was born in Budapest, Austria-Hungary. Wigner came to the US in 1930 and was Thomas D. Jones Professor of Mathematical Physics at Princeton University from 1938 until his retirement in 1971. He was educated in Europe and obtained his doctoral degree at the Technische Hochschule Berlin. He had a son and a daughter, one of them was a professor of mathematics at the University of California, Berkeley. During World War II, Wigner was actively involved in the Manhattan Project. After the war, Wigner was active as a member of many prestigious scientific organizations, and was the vice-president, and later, president, of the American Physical Society. A prolific physicist, Wigner left his mark in many areas in physics and mathematics. Many theorems and phenomena are named after him, including Wigner crystallization, the Jordon-Wigner transformation, and the Wigner theorem.

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